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Non-Euclidean geometries in and for the mathematics teacher education: a literature review

Geometrias não euclidianas na e para a formação de professores de matemática: Uma revisão da literatura

Las geometrías no euclidianas en y para la formación del profesorado de matemáticas: Una revisión de literatura

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Abstract: Science challenges the relevance of school geometry in its treatment of space. Therefore, the research discusses the incorporation of non-Euclidean geometries in the initial and continuing education of mathematics teachers. Hence, this literature review aims to synthesize and describe, in a broad and current range, the recent state of research in educational mathematics on these geometries in teacher education. To do this, we configured a method in three moments: the search, evaluation, selection, and organization of the sources found and reading, analyzing, and writing results. The results highlight a regional and temporal interest that prioritizes teaching practice. Furthermore, a deficit and prescriptive perspective of teachers' knowledge and a minority perspective of the teacher as a subject of knowledge are recognized. We conclude with the need for research on various phenomena, including Euclidean generalizations.
Keywords: Mathematics education. Contents of teacher education. Geometry.

Resumo: A ciência questiona a relevância da geometria escolar por seu tratamento do espaço. Portanto, a pesquisa discute a incorporação de geometrias não euclidianas na formação inicial e continuada de professores de matemática. Assim, esta revisão de literatura visa sintetizar e descrever, de forma ampla e atual, o estado recente da pesquisa em educação matemática sobre essas geometrias na formação de professores. Para isso, configurou-se um método em três momentos: a busca; a avaliação, seleção e organização das fontes encontradas; e leitura, análise e redação dos resultados. Os resultados destacam um interesse regional e temporal que prioriza a prática docente. Além disso, reconhece-se uma perspectiva deficitária e prescritiva do conhecimento dos professores e uma perspectiva, minoritária, do professor como sujeito do conhecimento. Conclui-se que há a necessidade de pesquisas sobre diversos fenômenos, inclusive generalizações euclidianas.
Palavras-chave: Educação matemática. Conteúdos da formação de professores. Geometria.

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Resumen: La ciencia cuestiona la pertinencia de la geometría escolar por el tratamiento que hace del espacio. Por ello, en la investigación, se discute la incorporación de geometrías no euclidianas en la formación inicial y continua del profesorado de matemáticas. De ahí que esta revisión de literatura se proponga sintetizar y describir, en un rango amplio y actual, el estado reciente de la investigación en matemática educativa sobre estas geometrías en la formación docente. Para ello, se configuró un método en tres momentos: la búsqueda; la evaluación, selección y organización de las fuentes encontradas; y la lectura, análisis y redacción de resultados. Como resultados, destaca un interés regional y temporal que prioriza la práctica docente. Además, se reconoce una perspectiva deficitaria y prescriptiva del conocimiento del profesorado y una perspectiva, minoritaria, del profesor como sujeto de conocimiento. Se concluye con la necesidad de investigación sobre varios fenómenos, entre ellos, las generalizaciones euclidianas.
Palabras-clave: Educación matemática. Contenidos de la formación docente. Geometría.

Introduction

Euclidean geometry has been the thematic foundation of school geometry since the beginning of the study of geometry since it allows organizing and systematizing intuitive and experimental knowledge of the space in which we live (Moreno-Armella; Brady; Elizondo, 2018). However, research in geometry education in recent years reflects the questions raised by science about the relevance of this knowledge for today's citizens due to the importance of visuospatial reasoning and the multidisciplinary treatment of space (Sinclair *et al*., 2016; Sinclair; Cirillo; Villiers, 2017; Viveros, 2019).

Current science is interested in understanding space and its changes, which is why scientific subjects are created and combined –such as astronautics, geonautics, geophysics, oceanic and naval engineering, space physics, and atmospheric sciences– that aim for a more detailed description of space and where Euclidean geometry is limited to the local (Bruce *et al*., 2017; Soares *et al*., 2020). Since the high school period, these sciences demand geometric knowledge that transcends from the local to the molecular and astronomical space to cement in the student body their curriculum knowledge and multicurriculum knowledge in university studies and, as citizens, allow them to deal with their environment.

Therefore, to develop reasoning and skills in that direction, non-Euclidean geometries (NEG) were incorporated into high school curricula in some places, with mathematics teachers responsible for ensuring their attention, which also involved integrating NEGs in curricula for initial mathematics teacher education and continuing education programs. However, in Costa Rica, Colombia, Chile, and Argentina, NEGs are not present at basic levels but are found in some curricula for the initial education of mathematics teachers.

With a background in geometry education, as far as teachers' geometric knowledge is concerned, NEGs in mathematics teacher education raise multiple worries. As significant emphasis has been placed on mathematical knowledge, some studies identify shortcomings, deficiencies, errors, and limited understandings (Jones; Tzekaki, 2016); that is, an approach focused on teachers' deficiencies

regarding particular mathematical processes or concepts (Ponte; Chapman, 2006), which materializes into prescriptions of what one should know. From this, we assumed that the approach to NEGs as mathematical knowledge for teaching would be addressed from these (deficient and prescriptive) perspectives.

However, in the field of teacher professional development, and specifically the research programs on the knowledge of mathematics teachers, we recognize that it is complex to refer to it, given the –necessary– combination of multiple expertise; therefore, for their description and structure, some researchers propose models to systematize them. For example, with some of these models, NEGs as mathematical knowledge must be intertwined with other acquirements, such as curriculum and pedagogical knowledge, among others. Therefore, incorporating these geometries in teacher education requires foundations in different directions.

In Brazil, for example, these geometries were incorporated in high school and teacher education. For this reason, research on NEGs generated at least two changes in the objects of study – not yet reflected in the global and general discussions on geometry education: from focusing on high school or basic education students to focusing on teachers and their education and from analyzing NEGs as mathematical knowledge to also studying its didactics. In recent years, concern about the study of NEGs in teacher education has increased, leaving with it research results that address its treatment, results that require connecting, and systematization as a product of the current state of recent research in this field.

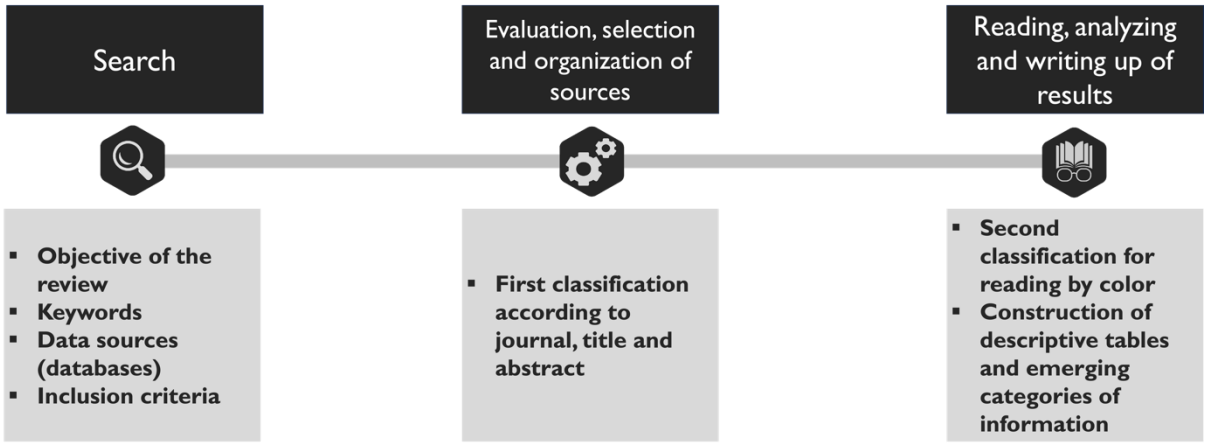
Therefore, this study proposes a literature review to synthesize and describe, in a wide range, the current state of research in educational mathematics on NEGs, considering the two main mathematics teachers' knowledges: hyperbolic geometry and elliptical-spherical geometry. With a global vision of research on NEGs as knowledge of mathematics teachers, this review will contribute to the field by synthesizing a comprehensive and current range of relevant studies on NEGs to teacher education. It also contributes to contextualizing and/or supporting new studies with said population in geometry area and showing open research spaces where interested researchers can recognize study phenomena.

Methodology

Based on the search, appraisal, synthesis, and analysis (SALSA) framework proposed by Grant and Booth (2009), for this literature review, we configured an empirical method with a degree of systematicity that could allow us to achieve the proposed objective. Although this framework was used by Grant and Booth (2009) to classify literature reviews, we took up their structure to constitute a

method in three moments: the search, evaluation, selection, and organization of the sources found; and reading, analyzing, and writing results (see Figure 1).

FIGURE 1. Literature review method



Source: Own construction.

Before the search, we needed to specify the objective of the review (presented above). In addition, keywords and their combinations were defined in Spanish, Portuguese, and English, as follows: "teacher (professor)" AND "geometry", "teacher (professor)" AND "non-Euclidean geometry," "teacher (professor)" AND "spherical geometry," "teacher (professor)" AND "hiperbolic geometry." Afterward, we determined the search locations, for which, given the possibility of access, the following databases and search engines were used: Dialnet, SpringerLink, JSTOR, Scopus, Scielo, Eric, Conricyt, Web of Science, Latindex, and Google Scholar.

Finally, we established inclusion criteria, considering any thesis (master's and doctorate), research article and/or book or book chapter where NEGs (hyperbolic geometry and spherical geometry) were investigated with mathematics teachers or students at any educational level, with references or reflections on teachers; and any document published from 2010 to April 2023, except for documents published in previous years, cited in the selected documents, which were within the time range and that are considered necessary for the project, that is, where NEGs are thought of as mathematics teachers' knowledge.

Therefore, the search was developed based on the elements above. Then, we evaluated, selected, and organized the sources found; for this, by reading their titles, abstracts, and journals to which they belonged, they were classified by journal and into two content categories as the first organization: studies on NGEs with mathematics teachers as the population, and studies with other

populations, students at different educational levels, that refer to teachers in their problems, discussions, and/or perspectives.

Then, to read the documents internally, in the two categories of the initial organization, the documents were classified and labeled by color: green, blue, and yellow. The documents labeled in green were articles or chapters with a structure of research report, theoretical, or historical reflection, without considering books or theses, which, in their entirety, deal with the NGEs and mathematics teachers' education. We read the sources marked in green in depth, prepared a bibliographic file for each, and made quick notes, reminders of ideas associated with the readings; literature notes, opinions, or reflections on specific sections of the reading; and permanent notes, reflections justified by arguments from various sources.

The documents labeled with blue contain only some sections that refer to the NEGs and mathematics teachers' education. Here, the theses, books, and some articles were contemplated. Only the sections of interest were read in depth, and brief literature and permanent notes were made about their information. Finally, the documents labeled with yellow did not focus on NEGs as teaching knowledge, but some sections, such as the problem, discussion, and conclusion, present reflections on it. The readings of the sections of interest from the sources marked in yellow were exploratory, and we took some brief notes from them.

From the products of the readings, in the analysis, we constructed descriptive tables of the documents considering the database, journal, year of publication, geometry they deal with (hyperbolic, spherical, or both), population, language, type of academic production (thesis, book, chapter, or article), country where the study was developed, and type of research (empirical or theoretical). Furthermore, in concept maps, emerging categories of information were built and organized, which allowed the objective to be met. Given its descriptive nature, an inductive method was chosen to construct these categories (MORAES, 2003), starting by asking what these investigations intend, seek, or explain.

We identified that the research exposes the importance of these geometries as knowledge of mathematics teachers. However, it recognizes some obstacles to studying these geometries in teacher education and, consequently, promotes didactic resources that can favor their treatment with this population. From this approach structure, we constructed the following categories of information: arguments that precede the incorporation of NEGs in the education of mathematics teachers, that justify or favor this incorporation, and the didactic elements proposed for said incorporation.

Results

The results of this review are presented in two phases: the descriptive phase, where a presentation of the selected documents is intended through some characteristics and their implications, and the analytical phase, the product of the detailed study of the emerging categories of information.

a) Descriptive phase

From the search, we took 38 documents as units of analysis, whose descriptions can be reviewed in Appendix A. When organizing them, they were classified into two initial categories of content: studies on NGEs with mathematics teachers as the population (28 documents) and studies with other populations, students at different educational levels, that refer to the teaching population in their problems, discussions, and/or perspectives (seven documents). However, since some documents did not apply to this classification, the documents that report literature reviews or theoretical reflections (three documents), we created a third category called *No population* (see Table 1).

Most of the documents are research articles because, typically, they condense the main results of an investigation. Furthermore, due to the classification by color, we read only some sections of the theses; on the other hand, the articles that had mathematics teachers as their population, which were the majority, were classified in green and were read in depth. The inclusion criteria specified that previously published documents that particularly discussed NGEs as mathematics teachers' knowledge could be incorporated. One of them was published in 2003.

TABLE 1. Particular characteristics of the studies

(continue)

Variables	Teachers	Students	No population	Total Quantity (%)
Language				
▪ Spanish	1	0	0	1 (2.6%)
▪ English	5	3	0	8 (21.1%)
▪ Portuguese	22	4	3	29 (76.3%)
Academic production				
▪ Doctoral thesis	1	0	0	1 (2.6%)
▪ Master's dissertation	2	0	0	2 (5.3%)
▪ Book chapter	0	2	0	2 (5.3%)
▪ Article	25	5	3	33 (86.8%)

(conclusion)

Country of the study				
▪ Canada	1	0	0	1 (2.6%)
▪ United States	1	0	0	1 (2.6%)
▪ Indonesia	1	0	0	1 (2.6%)
▪ Italy and Hungary	1	0	0	1 (2.6%)
▪ Italy	0	1	0	1 (2.6%)
▪ Mexico	1	0	0	1 (2.6%)
▪ Portugal	1	0	0	1 (2.6%)
▪ Chile	0	1	0	1 (2.6%)
▪ Greece	0	1	0	1 (2.6%)
▪ Brazil	22	4	3	29 (76.3%)
Type of research				
▪ Theoretical	7	0	3	10 (26.3%)
▪ Empirical	21	7	0	28 (73.7%)
Publication year				
▪ 2003, 2010-2011	2	0	0	2 (5.3%)
▪ 2012 - 2014	7	1	0	8 (21.1%)
▪ 2015 - 2017	8	1	1	10 (26.3%)
▪ 2018 - 2020	4	3	0	7 (18.4%)
▪ 2021 – (April, 2023)	7	2	2	11 (28.9%)
Color classification				
▪ Yellow	1	7	1	9 (23.7%)
▪ Blue	7	0	1	8 (21.1%)
▪ Green	20	0	1	21 (55.3%)

Source: Author's own creation.

This table allows us to note the situational interest of research in educational mathematics on NGEs as mathematics teachers' knowledge. Both the publication language and the country where the studies were developed demonstrate Brazil's interest in researching and discussing the topic (76.3%). However, we must specify that although the percentage is the same for the language and country, it does not refer to the same documents, given that there is one research developed in Brazil and written in English, and one research developed in Portugal and written in Portuguese. As mentioned in the introduction, Brazil has incorporated these geometries in primary and secondary school education in different states, so it makes sense to expand this incorporation and research in teacher education.

The years of publication show a temporary interest in this problem. Research on NEGs as mathematics teachers' knowledge has increased recently. Given that most of the research has been conducted in Brazil, we can think that the concern about this problem is situational in terms of space and time; in fact, the descriptive evidence of this review justifies this idea. However, before drawing

any conclusion, we should consider two more arguments: the first, that there are investigations in other countries that expose this concern and that do research on it, but that do not have the same justifications and motivations; and the second, that there are some countries that do not have NEGs incorporated in basic education, but do have them in the curricula for mathematics teacher education.

The characteristics of the selected documents represent a context of analysis, which will allow the recognition in the analytical phase of results based on particular experiences. With a second classification by color, we read the documents and structured the analytical phase based on the analysis categories.

b) Analytical phase

The results of the analytical phase are presented from the emerging categories of information, configured with the reading and critical analysis of the sources when one asks about the interest of these investigations.

Category 1: Arguments prior to the incorporation of NEGs in the education of mathematics teachers

These lines of reasoning address different situations. Therefore, for explanatory purposes, they were categorized into historical-educational arguments, recurring in educational research over time; cognitive arguments, referring to the learning process of these geometries; didactic arguments, associated with the didactic conditions and knowledge necessary for their study; and contextual arguments, linked to the social and cultural circumstances in which they are studied or could be studied.

Among the historical-educational arguments, we recognize two problems associated with school geometry: the poor updating of both geometric knowledge and didactic-geometric knowledge (Moreno-Armella; Elizondo, 2017; Santos, 2020) and pointing out geometry relegated by arithmetic and algebra and subject to being taught under their approaches (Ferreira; Oliveira; Dante, 2016; Pinto; Carneiro, 2011; Santos, 2020). In school geometry, "an exclusively axiomatic approach to geometric thinking continues to take root, in which the teacher prioritizes arithmetic and algebraic aspects to the detriment of geometric ones" (Santos, 2020, p. 59, own translation), which leaves geometry in the background. These problems hinder the incorporation of these geometries at any educational level since this incorporation would imply a radical change in school geometry and a revaluation of geometry.

In this same category of arguments, Gaiowski and Bassoi (2014) and Lovis, Franco, and Barros (2014) describe the belief that teachers, when studying NEGs, will manifest the same difficulties as those who built them. When citing the obstacles and resistance that made the construction of NEGs

difficult, Lovis, Franco, and Barros (2014, p. 20, own translation) mentioned that "the hypothesis is that these difficulties, these obstacles, also manifest themselves in the teachers/students when they study these contents." Now, it is essential to note that the construction of these geometries has not been linear; it has been transformed with the contributions of many people and multiple sociocultural circumstances. In addition to this idea, the mathematics intended to be part of teachers' knowledge has gone through transposition processes enhanced by political, commercial, and cultural interests and research results, among others, which justifies the notion that this knowledge has driven away from the problems, contexts, and difficulties faced and used by those who built them.

Among the most exposed arguments, cognitive arguments stand out. The first refers to the instability of the teachers' Euclidean knowledge; that is, given that in some cases they do not know, combine, or use certain Euclidean geometric notions interchangeably, it is complex to think about the study of new geometries (Lovis; Franco; Barros, 2014; Caldatto; Pavanello, 2014; Pereira; Manrique; AntuneS, 2022). However, more research is required on this instability to refer to its treatment.

Another obstacle associated with Euclidean geometry is Euclidean generalizations, also called Euclidean ideas or rules. According to Lovis, Franco, and Barros (2014), it is evident when some Euclidean reasoning –valid only in this geometry– can be considered general without recognizing the specificity of the surface on which one works, for example, the sum of the internal angles of a triangle is equal to 180°, a single parallel passes through a point outside a line, and the triangle is the polygon with the fewest number of sides. According to Assis (2017), Pinto (2013), Rocha *et al*. (2020), and Santos (2020), these generalizations are a consequence of the unique permanence of Euclidean geometry in the geometric knowledge of teachers. The roots of these generalizations can take precedence over learning other geometries; however, the exposure of their limitations on other surfaces generates immediate differences between geometries and, with them, questions about particular geometric notions.

In this same category of arguments, the presentation and representation of the NEGs stand out. According to Assis (2017), in the few places where NEGs in teacher education are studied, these geometries are presented from a formalist approach; that is, under an axiomatic system so that teachers distance themselves and, consequently, decide to discriminate against their teaching. Likewise, the graphic representations of geometric objects in these geometries are hardly related to that object. We can recognize a justification in Lovis, Franco, and Barros (2014, p. 28, own translation), who affirm that "teachers believed that the representations of geometric objects were, in fact, the object itself," which makes it difficult to associate two representations with the same object.

Finally, among the cognitive arguments, linguistic aspects also take precedence. Given the focus on Euclidean geometry, when studying other geometries, teachers interact with terms, rules, or axioms that *a priori* lack meaning, just like the biangle and the trirectangular triangle in spherical geometry (Lovis; Franco; Barros, 2014). These arguments make the didactic aspect in teacher education more complex, hence the didactic arguments are focused on deficiencies.

Regarding didactic arguments, Aparecida (2015), Caldatto and Pavanello (2014), Ferreira, Oliveira, and Dante (2016), and Gambini and Lénárt (2021) recognize a lack of disciplinary and didactic knowledge in teacher educators associated with the NEGs. This argument comes from the expansion of the incorporation of NEGs. For example, in some states of Brazil, NEGs were first incorporated into high schools. From there, they had to be integrated into teacher education, which means that teacher educators were also required to know them (Caldatto; Pavanello, 2014).

Furthermore, to justify the study of these geometries in teacher education, there are still deficiencies in didactic resources and teaching methodologies (Aparecida, 2015; Gaiowski; Bassoi, 2014; Santos, 2020). Teachers from Paraná, Brazil, who participated in incorporating NEGs in high schools allude to the lack of textbooks and materials that favor the treatment of these geometries (Caldatto; Pavanello, 2014). As mentioned by Gaiowski and Bassoi (2014, p. 2, own translation), the incorporation of NEGs becomes complex, given "the lack of knowledge of the content, the lack of methodological guidelines by a large part of the teachers, or even the absence of textbooks addressing the topic."

The population participating in the study by Caldatto and Pavanello (2014) also exposed the lack of time to attend to an expanded curriculum, given that incorporating these geometries only increased the content. Furthermore, given these conditions, which have always been a concern, one of the topics that will probably be excluded from teaching will be NEGs.

The last arguments in this section are associated with the circumstances of the context in which we intend that teachers study these geometries. The first can be recognized in any population that studies these geometries: the influence that the local context has on people's rationality (Pinto, 2013). The local context is well described by Euclidean geometry, therefore, the study of new geometries represents a great challenge, the challenge of thinking in astronomical and molecular contexts. Furthermore, in this category of arguments, Caldatto and Pavanello (2014) and Pereira, Manrique, and Antunes (2022) recognize that the adversities faced by public education hinder any development in education, consequently, also the incorporation of NEGs in teacher education. Caldatto and Pavanello (2014) argue that the precarious conditions of teaching professional development, institutional infrastructure, and research faced by public education in Brazil predispose the consideration of new mathematical content.

Likewise, the role of mathematics teachers in curricular changes has repercussions on the study of these geometries (Caldatto; Pavanello, 2014; Pereira; Manrique; Antunes, 2022). Given their professionalism and recognition as subjects of knowledge, they should be able to participate in the incorporation of these geometries actively, since only with their participation will this incorporation occur in practice. Caldatto and Pavanello (2014) point out that in Brazil, teachers were cajoled into highlighting a discourse of collective construction of curriculum guidelines without reflecting on these geometries and the implications of their incorporation.

The arguments preceding the incorporation of these geometries in teacher education emerge from the experiences and realities of some places, from the necessary research on some phenomena related to these geometries, and from the appreciation of geometry and mathematics teachers. The same researchers who argue in this sense respond to their arguments by substantiating said incorporation. As a consequence, the second category of analysis emerges: the arguments that justify or favor the incorporation of NEGs in mathematics teacher education.

Category 2: Arguments that justify or favor the incorporation of NEGs in mathematics teachers' education

Just like the cons, the pros of incorporation were categorized into historical-epistemological arguments, which come from the value of historical data and the construction of these geometries; cognitive arguments, referring to the acquisition of this knowledge; didactic arguments, associated with didactic conditions and knowledge; and contextual arguments, linked to social and cultural circumstances.

Among the historical-epistemological arguments, many researchers say that the study of these geometries allows teachers to reflect on the conception of space, mathematical truth, rigor, consistency, and axiomatic systems (Aparecida; Pinto, 2021; Assis, 2017; Caldatto; Pavanello, 2014; Santos, 2020; Soares *Et Al*., 2021; Song; Schwenz, 2013). Therefore, this generates discussions about the construction of science and mathematics evolution. The consistency of NEGs justifies this reflection; for example, Assis (2017, p. 396, own translation) states that "the consistency of hyperbolic geometry makes the debate about its inclusion in the curriculum of mathematics teachers' formative courses plausible," in other words, that the veracity of this knowledge attributes to it a potential to be known at least by mathematics teachers.

Furthermore, these geometries enhance teachers' historical and philosophical knowledge by analyzing their construction and the sociocultural circumstances that allowed it (Soares *et al*., 2020). The Political Pedagogical Project (PPP) of the state of Paraná in Brazil exposes the relevance of teachers'

history knowledge on the mathematical basis of the NEGs by denying Euclid's fifth postulate (ASSIS, 2017). Consequently, the study of these geometries generates in teachers a constructible and in-construction conception of mathematics (Aparecida; Pinto, 2021; Pivatto; Schuhmacher, 2013; Pivatto; Schuhmacher; Silva, 2016), which impacts positively and directly on their teaching practice, by presenting to their students "the construction of mathematical knowledge as something dynamic, alive, transformative, to deconstruct the vision of a ready-made science, of untouchable results, of a unique knowledge" (Santos, 2020, p. 47, own translation).

In turn, among the cognitive arguments is the most prominent one regarding the study of NEGs in any population, the potential of these geometries in the significance of Euclidean geometry (Gambini, 2021; Lovis; Franco, 2015; Oliveira; Ferreira, 2020; Santos, 2020; Santos; Souza, 2021a, 2021b; Souza, 2022; Wasserman; Stockton, 2013). When teachers reflect on geometric notions on surfaces other than the plane, they question those notions in the plane and, with this, attribute new meanings to it. In addition, they rethink the didactic aspects of both Euclidean and non-Euclidean geometry (Oliveira; Ferreira, 2020).

Referring to teaching NEGs through the approach of comparison between geometries, Gambini (2021, p. 242, own translation) concludes that "the teachers recognized the advantages of this method, not only to awaken and reinforce interest in geometry but also to improve and accelerate the understanding of Euclidean concepts." The justifications for studying NEGs are not intended to discriminate against Euclidean geometry; in fact, this is recognized as the essence of rationality, as it favors dealing with local space.

Likewise, in cognitive arguments, these geometries favor a geometric culture (Coelho; Perovano; Ribeiro, 2020; Kotarinou; Stathopoulou, 2017; Pinto, 2013; Pinto; Carneiro, 2011; Pinto; Portella; Machado, 2017), a mathematical culture (Pinto; Carneiro, 2011) and a scientific culture in teachers (Kotarinou; Stathopoulou, 2017; Pinto; Carneiro, 2011). Referring to the possibility that the sum of the internal angles of a triangle can be bigger or smaller than 180°, Pinto and Carneiro (2011, p. 91, own translation) mention that "questions like these develop mental processes in students that lead them to seek to improve their geometric culture during their education." The unique knowledge of Euclidean geometry limits teachers' teaching on the treatment of space. Furthermore, the construction of the NEGs represents a form of scientific knowledge construction, the development of knowledge from denying a previously established one.

We present didactic arguments very close to the previous ones. As mentioned above, given that NEG knowledge enhances the significance of Euclidean geometry, Oliveira and Ferreira (2020) and Santos (2020) state that teachers' knowledge of these geometries contributes to the quality of teaching geometry in primary and secondary education, given that their study can be seen as a "possibility of

changing how teachers understand and carry out geometry teaching" (Santos; Souza, 2021a, p. 16, own translation). Likewise, in places where NEGs are part of the curriculum guidelines at that level, this incorporation allows teachers to respond to curriculum and institutional demands upon entering the labor system (Pinto, 2013).

In turn, among the contextual arguments, NEG knowledge favors the fulfillment of the objective attributed to geometry, the development of thinking that allows us to deal with our environment (Assis, 2017; Kotarinou; Stathopoulou, 2017; Santos, 2020; Santos; Souza, 2021a). According to Santos (2020, p. 198, own translation), by accepting the existence of other geometric models, teachers "modify the way they relate to the space in which they live and, thus, develop greater autonomy in the construction process of the geometric concepts under study." Knowledge of the NEGs allows scientists to explain the physical world by recognizing atomic, local, and astronomical space. Santos (2020) states this recognition would raise awareness of teaching practice in geometry, particularly the treatment of space.

In addition, responding to this objective implies assessing the demands of current science, which requires a detailed description of space, strengthening interdisciplinarity in teacher education (Pivatto; Schuhmacher, 2014; Santos; Souza, 2021b). On the other hand, when they interact with NEGs, they recognize their role in the explanation of space and accept the limitations of Euclidean geometry when representing some physical phenomena (Kotarinou; Stathopoulou, 2017; Oliveira; Ferreira, 2020; Santos, 2020; Santos; Souza, 2021a; Soares *et al*., 2021).

The arguments against the study of these geometries by mathematics teachers are associated with:

- school geometry, its updating, and the role that geometry plays;
- the construction of the NEG, its presentation and representation, and
- the lack of specific and didactic knowledge.

Rather than taking precedence over teachers' studies of these geometries, they are arguments that expose the necessary foundation and justification for incorporating these geometries based on research.

From their part, the pros respond to the why and wherefore of NEGs in teacher education. The evidence shows that the justification for the study of the NEGs not only comes from the fact that these geometries are in the curriculum guidelines of primary or secondary education, where the teachers will work; many arguments aim to generate broader geometric, mathematical, and scientific knowledge in the teaching population, for the benefit of their teaching practice, even if it is focused on Euclidean geometry. Although we cannot assure that there are sufficient elements to justify and

substantiate teachers' study of these geometries, some didactic proposals that contribute in this sense can be recognized, and the third category of analysis is intended to describe them.

Category 3: Didactic elements proposed to carry out said incorporation

Given that the research proposes different didactic elements, its report was organized into introductory models, understood as geometric contexts agreed upon for the study of these geometries; didactic strategies, which refer to particular processes associated with methodological approaches; and methodological approaches, which generally structure the teaching and learning processes of this geometry in teacher education.

Among the introductory models associated with hyperbolic geometry, the Klein circle stands out –a circle in which the chords are considered straight lines (Assis, 2017; Rocha *et al*., 2020); the Poincaré circle –where the straight lines are arcs of circles orthogonal to the border (Assis, 2017; Kotarinou; Stathopoulou, 2017); and the Poincaré semiplane; as described by Moreno-Armella and Elizondo (2017), if a straight line crosses the Euclidean plane, then the Poincaré semiplane is one of those semiplanes, where the semicircles that are orthogonal to the boundary straight line are considered straight lines.

In turn, one of the most common considerations in spherical geometry is the surface of the sphere; for this, different spheres can be used as tools. Among the most notable is the Lénárt sphere, proposed by the Hungarian professor and researcher István Lénárt, which has its own construction and measurement tools: spherical sheets, spherical compass, and spherical ruler (Gambini, 2021; Gambini; Lénárt, 2021). Pinto (2018) proposes a second model for spherical geometry: stereographic projection, understood as the representation of the surface of the sphere by projecting all its points from one of them, normally the north pole in a unit sphere, onto the horizontal plane that passes through its center.

Regarding the teaching strategies used and recommended from the research, given that they are relatively new contents, Ferreira, Oliveira, and Dante (2016), Gaiowski and Bassoi (2014), Gambini and Lénárt (2021), Oliveira and Ferreira (2020), and Song and Schwenz (2013) allude to the need to teach through research results that substantiate what and how. Regarding teaching processes, the data sources suggest various strategies, some associated with disciplinary knowledge and others related to didactic knowledge. Regarding the strategies linked to disciplinary knowledge, Pinto (2013), Pinto (2018), and Pinto, Portella, and Machado (2017) suggest an initial treatment in an exploratory manner, that is, with a gradual introduction of its axiomatics.

Regarding didactic knowledge, teaching is proposed through interdisciplinarity and contextuality, in other words, through different application contexts that favor the approach to these geometries (Aparecida; Romualdo, 2015; Caerols; Carrasco; Asenjo, 2021; Conceição, 2018; Pataki, 2003; Pinto; Portella; Machado, 2017; Song; Schwenz, 2013); for example, for spherical geometry, some of these contexts are: geography (Conceição, 2018; Pataki, 2003; Pinto, 2013), art-photography (Caerols; Carrasco; Asenjo, 2021), air and maritime navigation (Aparecida; Romualdo, 2015; Pinto, 2013), geology —mainly in paleomagnetism that studies the Earth's magnetic field— (Aparecida; Romualdo, 2015), Astronomy (Aparecida; Romualdo, 2015; Caerols; Carrasco; Asenjo, 2021), mathematics itself (Pinto; Carneiro, 2011), and the Global Positioning System (GPS) (Song; Schwenz, 2013).

For their part, Assis (2017), Ferreira, Oliveira, and Dante (2016), Oliveira and Ferreira (2020), and Silva and Costa (2022) recognize the potential of the history of NGEs in teaching these geometries. Alluding to history for didactic purposes for the study of NGEs, Oliveira and Ferreira (2020, p. 478, own translation) state that "it is possible to think of a teacher education that explores the concepts that compose school mathematics, but that also allows a pedagogical look at how to teach these concepts." In addition to these strategies, Gambini (2021) maintains that in the characterization of mathematical notions, not only of NEGs, movement and experiences play an essential role since, through the use of the body and its senses, learners organize the doing and guide their perception; hence, the author proposes sensory-motor experimentation as a strategy.

One of the didactic routes for teaching Euclidean geometry is by using dynamic geometry software, a strategy that transcends the teaching of NEGs in teacher education (Assis, 2017; Ferreira; Oliveira; Dante, 2016; Kotarinou; Stathopoulou, 2017; Moreno-Armella; Elizondo, 2017; Pinto, 2018; Rocha *et al*., 2020; Soares *et al*., 2020; Song; Schwenz, 2013). Some of the proposals for these geometries are Cabri Geometry, Cinderella, GeoGebra, No Euclid, and Cabri II. According to Ribeiro and Perovano (2020), although the potential of dynamic geometry environments in assisting the NGEs is recognized, modifications and conventions are suggested since the software is programmed under Euclidean rationality.

The strategy suggested in the initial phase of the NEGs study is the use of manipulative materials (Caerols; Carrasco; Asenjo, 2021; Conceição, 2018; Gambini, 2021; Kotarinou; Stathopoulou, 2017; Pivatto; Schuhmacher, 2013; Soares *et al*., 2020) since, by involving the senses of sight and touch, learning these geometries becomes more significant.

These strategies are not promoted independently. As a whole, they aim to move from the experimental to the abstract and the move from the manipulative to the digital. In this sense, Gambini

(2021, p. 241, own translation) states: " We believe that the most important result obtained has been the multimodality of teaching, the contemporary use of mathematical and laboratory tools in class." Multimodality refers to using various didactic strategies in teaching these geometries (Kotarinou; Stathopoulou, 2017).

Regarding methodological approaches, three are recognized. The first is teaching the NEGs using a comparative approach between one or several of them and the Euclidean geometry. Proposed in 1990 by István Lénárt in Hungary (Gambini, 2021; Gambini; Lénárt, 2021; Pivatto; Schuhmacher, 2014), it originally consisted of comparative and contrasting teaching between plane geometry and spherical geometry; however, the latest works by Lénárt and his group proposed adding hyperbolic geometry to it (Gambini; Lénárt, 2021), so the contrast would occur between the three geometries.

The second is the competency- and skill-based approach proposed by Pivatto and Schuhmacher (2014) in Brazil, particularly for teaching NEGs in initial teacher education. This approach seeks to develop pre-service teachers' academic competencies, such as simplifying NEGs, understanding their logical structure, recognizing relevant variables such as geodesics, and establishing incompatible relationships between geometries. It also seeks to develop didactic skills, such as using prior knowledge, reinforcing Euclidean geometry, and being clear about the objective of studying NEGs. Finally, the ethnomathematical approach, proposed by Sukestiyarno *et al*. (2023) in Indonesia, aims to develop spatial skills by studying NEGs, making mathematics more relevant and meaningful through real application contexts.

These didactic elements seek to show the routes of attention of NEGs in teacher education. However, questions arise about their relevance due to the manner or degree of attention given to the problems, which will be discussed in the following section.

Discussion

For an in-depth reflection on some particular points, we developed this section into four moments: first, we recognize the implications of the characteristics of the reviewed documents; then, we reflect on the elements that influence teachers' study of NEGs; next, the contributions of these elements to the problems discussed in the field of NEG education are presented; and, finally, the particular potential of this literature review.

Regarding the first, given the regional and temporal interest of the research presented in the descriptive results, we recognize a context in which the results of such studies refer to a demand; i.e., in some states of Brazil, NEGs are part of the high school curriculum guidelines, so support for teaching training and practice is a priority. The analytical results recognize arguments that justify the study of

these geometries in teacher education and are not associated with their teaching, such as the potential of NEGs in the significance of Euclidean geometry. These arguments are other reasons for studying NEGs in mathematics teacher education.

The studies developed in Hungary and Italy reflect that this object of study has different kinds of justifications, which give grounds to the thought that in some countries, NEGs are part of teacher education without being part of the high school curriculum guidelines and basic education. In this regard, some of the elements that influence the study of these geometries are seen to come from school geometry. The lack of updating of school geometry and the fact that it has become a context of application for arithmetic and algebra has caused instability in Euclidean knowledge and the constitution of Euclidean generalizations in teachers (Santos, 2020), phenomena that affect the study of new geometries. On the other hand, the study of NEGs strengthens school geometry because it favors the significance of Euclidean geometry and contributes to the quality of its teaching (Gambini, 2021; Santos; Souza, 2021a, 2021b; Souza, 2022).

Other elements come from the history of geometry. The construction of the mathematical foundation of NEGs represents a milestone in the history of mathematics and science. The denial of Euclid's V postulate became a method of constructing scientific knowledge (Pinto; Carneiro, 2011); it is an example that shows that mathematics is a human activity; it is alive, it is built and reconstructed over time (Aparecida; Pinto, 2021); a paradigmatic case that allowed truth, rigor, and mathematical consistency to be reconsidered (Soares *et al*., 2021); and represents the beginning of the construction of knowledge that rethinks the treatment with space (Oliveira; Ferreira, 2020).

The nature of the NEGs also influences teachers' study of them, as noted in the arguments about the representation of notions in these geometries and the linguistic aspects related to them (Assis, 2017). Furthermore, the teachers who were part of the process of incorporating these geometries in the state of Paraná, Brazil, exposed the need to delve deeper into their particular characteristics (Caldatto; Pavanello, 2014), showing a need for research focused on their nature. Considering their particularities, the epistemological knowledge associated with these geometries would provide reference on what and how to teach them. In addition to these elements, the contextual circumstances they face in education also affect their study, considering that these implications will be particular, depending on the region referred.

In the third moment, thinking about NEGs as mathematics teachers' knowledge implies recognizing their complexity given the necessary variety of knowledge. From the field of research in geometry education, we understand the emphasis on the questions raised by science on the relevance of school geometry concerning the treatment of space and the dominance of research approaches

focused on deficiencies and prescriptions on teachers' mathematical knowledge. Arguments about the instability of their Euclidean knowledge, the belief that when studying NEGs, they will manifest the same difficulties as those who built that mathematics, and the view of Euclidean generalizations and the representation and nomenclature of NEGs as obstacles are evidence of the deficient perspective of teachers' knowledge about geometric processes and concepts related to NEGs.

On the other hand, arguments that recognize the importance of teachers' roles in curriculum changes and that accept that the study of new geometries also depends on the adversities they face in the classroom and educational institutions, on the lack of time, resources, and teaching methodologies, are evidence of a vision of the teacher as a subject of knowledge. As Tardif (2014, p. 168-169) points out, this vision must be promoted through research on teachers based on a change in conception: from seeing them as "technicians who apply knowledge produced by others" to recognizing them as "subjects who possess, use, and produce specific knowledge of their profession," which implies accepting that in their work and education, they produce, reformulate, and energize knowledge.

Of the didactic elements, the multimodality of NEG teaching recognizes the professionalism of the teachers when selecting, according to their experience and context, the didactic strategies they consider appropriate. Furthermore, the approach based on competencies and skills (Pivatto; Schuhmacher, 2014) shows the necessary variety of knowledge associated with NEGs as teaching knowledge. However, it focuses on what teachers should know about them and their teaching.

Finally, this literature review enables the recognition of didactic phenomena that require further research, of which we highlight the causes and consequences of the instability of Euclidean knowledge in teachers, the limitations of Euclidean generalizations on surfaces other than the plane, the phenomenon of expansion of the incorporation of NEGs, their epistemology and history, and influence of the knowledge of NEGs in teaching Euclidean geometry. Furthermore, both the arguments and the didactic elements can support proposals for didactic intervention with teachers and students and contribute to the justification and contextualization of the incorporation of these geometries in mathematics teacher education.

Conclusions

Current research on NEGs in and for teacher education has been oriented toward qualitative empirical data, with a boost in recent years. Furthermore, its results are mostly in the Portuguese language. The experience of some places where NEGs are in the curriculum guidelines for teacher education shows a vision of the current state of the NEGs *in* that regard. Hence, the presentation, representation, and terminology about them have generated difficulties in their study and strategies,

and there have been founding studies on methodological approaches to their teaching. On the other hand, this review shows relevant information *for* incorporating or studying NEGS in teacher education, for example, recognizing the influence of Euclidean generalizations in their study and teaching resources as products of these investigations. Therefore, in this review, we refer to NEGs *in* and *for* teacher education.

In terms of the content of the research reviewed, a mastery of mathematical culture is evidenced by the recognition that the deficit and prescriptive perspective of teachers' knowledge – correspondingly, what they do not know and what they should know– described from the general field of geometry education, remain in the field of NEGs as mathematics teachers' knowledge. Although some studies consider teachers a subject of knowledge, given that they are a minority, more research is required to support this vision.

The reviewed research is focused on justifying the importance of NEGs as current citizens' knowledge of and mainly as teachers' knowledge, hence the construction of emerging categories of information. The pros and cons contribute to answering what, for what, and why NEGs in mathematics teacher education, and the didactic elements contribute to answering the how. However, none of these questions are fully answered, hence the importance of knowing the research progress with this interest.

The panorama of these studies externalizes challenges and phenomena of interest for research in the field. Some are associated with school geometry, for example, its relevance to current science; others are related to school geometry, with a direct influence on NEGs, among them Euclidean generalizations; and several are linked to NEGs, such as their particularities. Consequently, some questions arise: How do mathematics teachers construct non-Euclidean knowledge? What repercussions do their Euclidean knowledge have on this construction? What influence does knowledge of the NEGs have on their teaching practice? How does the study of NEGs favor the significance of Euclidean geometry? How can we go from manipulatives to digital and from experimental to abstract in didactic strategies for teaching NEGs?

As a consequence, the review shows the novelty of studies on NEGs as teachers' knowledge, the magnitude of these geometries in mathematics teacher education, and the current state of a research field that is consolidated based on the attention to school geometry problems. Therefore, this review contributes to geometry education, teacher professional development, and NEGs education. Moreover, it contributes particularly to researchers interested in the topic, educational authorities, and, mainly, mathematics teachers.

Appendix:

Appendix A – Particular characteristics of the selected documents:

https://doi.org/10.7910/DVN/IRSMYR

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